On local finite-time stabilization of the Viscous Burgers equation via boundary switched linear feedback

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Abstract— **This paper considers the problem of local finite-time stabilization of the viscous Burgers equation. A boundary switched linear control with state dependent switching law is designed based on the Backstepping approach. The strategy builds on discontinuous kernels which render the control function a piecewise continuous one. It is proved that such a control stabilizes locally the viscous Burgers equation and that the settling time depends on initial conditions. A simulation result is provided to validate the theoretical results.**

I. Introduction

For many dynamical systems, most of the results on stabilization and estimation are based on asymptotic or exponential guarantees. However, when the time of control is strongly restricted and transient process has to be finished in a finite-time, the need to address a finite-time or fixed time stabilization (or a very rapid stabilization, otherwise) becomes a central issue and necessity.

Finite-time concepts have been extensively considered in the framework of linear and nonlinear ordinary differential equations (ODES) ([10], [1], [19], [15], [22]). For partial differential equations (PDEs), finite-time concepts have been gaining a lot of attention as PDEs may indeed describe many complex systems (e.g. hydraulic networks, tubular chemical reactors, etc) for which the convergence while meeting time constraints or just realizing the well-known separation principle are central issues. Synthesis of controllers to achieve these goals would bring more challenges than exponential stabilization. For hyperbolic PDEs, for instance, one can refer to [16], [2] for the stabilization in finite-time with boundary control. For linear parabolic PDEs, on the other hand, the scenario is even more demanding. Some works have addressed some relevant issues on null controllability and finite-time stabilization (e.g. [3], [18]) by making use of the backstepping approach to design time-varying feedbacks; although with discontinuous kernels. It is worth recalling that the backstepping method has been used as a standard method to design boundary controllers for stabilization of PDEs [14].

This topic is moved forward for continuous time-varying feedbacks in [6] and [7] under backstepping approach for fixed-time stabilization. The time of convergence can even be prescribed in the design. Other results on finite-time for linear parabolic PDEs using in-domain control can be found in e.g. [17], [19] where the former uses some sliding mode techniques and the latter uses homogeneity arguments.

Nonlinear parabolic PDEs have been also useful for the modeling of large scale networks and complex processes coming from fluid mechanics. In particular, the viscous Burgers equation has been studied as a simplified model of the Navier-Stokes equations. The viscous Burgers equation can be in fact a tractable control-oriented model for which boundary and in-domain control strategies have been proposed e.g. [12], [11]. This motivates, from the practical point of view, to use this model for active flow control strategies to reduce the drag in aerodynamic applications and transportation industry. On the other hand, from theoretical level, this equation posses interesting challenges as sock-like stationary solutions may be developed (see e.g. [13]) and that have been useful for formation control problems in multi-agent systems; see e.g. [9]. Thus, stabilization and estimation of the viscous Burgers equation is of the great interest. However, to the best of our knowledge, finite-time stabilization and estimation for this kind of PDE have not been widely addressed in the literature. This motivate to study this topic under a boundary control problem formulation.

The main contribution of this paper relies on the design of a boundary switched linear control such that a suitable switching law is dependent on the state of the system. Inspired by [5] and [18], we build on the backstepping approach from which kernels of the transformation are piecewise-continuous. Under the switching law the control gain is such that it increases more and more while the solution of the closed-loop systems goes to zero in a finite-time. We prove that for sufficiently small initial condition, it is possible to achieve finite-time stabilization and that the settling time depends on initial data of the closed-loop system. We recall here that in this framework, finite-time convergence differs from the fixed-time one whenever the settling-time depends on the initial condition of the system as studied in this paper.

This paper is organized as follows. In Section II, we introduce the viscous Burgers and some preliminaries on the backstepping approach. Section III provides our approach towards fine-time stabilization. Section IV provides a

*This work has been partially supported by CPER ELSAT 2020, Project, Contratech and by Project 14.Z50.31.0031 of the Ministry of Education and Science of Russian Federation and Grant 08-08 of the Government of Russian Federation. N. Espitia is with Valse team, Inria Lille - Nord Europe, and CRIStAL (UMR-CNRS 9189), Ecole Centrale de Lille, Universit de Lille, Cité Scientifique, 59651 Villeneuve-dAscq, France. A. Polyakov is with Valse team, Inria Lille - Nord Europe, 40. av Halley, Villenueve dAscq, France, and with ITMO University, Saint-Petersburg, Russia. E. Fridman is with Department of Electrical Engineering and Systems, Tel Aviv University, Tel Aviv 69978, Israel Corresponding author: `nicolas.espitia-hoyos@inria.fr`

numerical example to illustrate the main results. Finally, conclusions and perspectives are given in Section V.

Notations: $\mathbb{R}^+$ will denote the set of nonnegative real numbers. The set of all functions $g : [0,1] \to \mathbb{R}$ such that $\int_0^1 g(x)^2 dx < \infty$ is denoted by $L^2((0,1),\mathbb{R})$ and is equipped with the norm $\|\cdot\|_{L^2((0,1),\mathbb{R})}$. $I_m(\cdot)$, $J_m(\cdot)$ with $m \in \mathbb{Z}$, denote the modified Bessel and (nonmodified) Bessel functions of the first kind, respectively.

II. Problem statement, preliminaries and basic assumptions

Let us consider the following viscous Burgers equation with Dirichlet boundary conditions:

$$u_t(t,x) = \theta u_{xx}(t,x) - u_x(t,x)u(t,x) \tag{1}$$
$$u(t,0) = 0 \tag{2}$$
$$u(t,1) = U(t) \tag{3}$$

and initial condition:

$$u(0,x) = u_0(x) \tag{4}$$

where $u : [0,T) \times [0,1] \to \mathbb{R}$ is the system state and $\theta > 0$ is the viscous term. In addition, $U(t) \in \mathbb{R}$ is the control input having the functional form

$$U(t) = \mathcal{K}_{\sigma(t)}[u(t,\cdot)](1) \tag{5}$$

with

$$\mathcal{K}_{\sigma(t)}[u(t,\cdot)](1) = \int_0^1 k(1,y,\lambda_{\sigma(t)})u(t,y)dy \tag{6}$$

which is the linear switching feedback law. Here, k will be a time-varying kernel gain (piecewise-continuous function) and $\sigma : [0,T) \to \mathbb{Z}$ is going to be a state dependent switching signal (piecewise-constant function), both to be characterized later on .

In open loop (e.g. $U(t) = 0$), the system (1)-(4) is globally exponentially stable. The control aim is to steer the state of the system (1)-(4) to zero in a finite-time. As it is going to be stated later, the time of convergence depends on initial conditions of the system.

A. Backstepping transformation and switching kernel equations

In this work we aim at providing a boundary linear switched control which is going to be designed based on the backstepping approach (inspired by [14], [11], [5] and [18]). The backstepping method has been typically applied to linear PDE systems. However, in this work the method may be applicable as soon as one considers the nonlinear term $u_x u$ in (1) as a small force (as typically done for the Kortewegde Vries equation, e.g. [23]). A characterization for the backstepping transformation as well as target system by taking into account the nonlinearity after transformation, is going to be discussed in the sequel. To that end, let us first bring back the following integral Volterra transformation as follows:

$$\begin{aligned} w(t,x) &= u(t,x) - \int_0^x k(x,y,\lambda_{\sigma(t)})u(t,y)dy \\ &= \mathcal{K}_{\sigma(t)}(t)[u(t,\cdot)](x) \end{aligned} \tag{7}$$

whose inverse is given by

$$\begin{aligned} u(t,x) &= w(t,x) + \int_0^x l(x,y,\lambda_{\sigma(t)})w(t,y)dy \\ &= \mathcal{L}_{\sigma(t)}(t)[w(t,\cdot)](x) \end{aligned} \tag{8}$$

where k and l are the time-varying kernels of the direct and inverse backstepping Volterra transformations, respectively.

Similar to [11], the aim is to transform the system (1)-(4) into the following target system:

$$\begin{aligned} w_t(t,x) = {} & \theta w_{xx}(t,x) - \theta\lambda_{\sigma(t)}w(t,x) \\ & -\mathcal{F}[w(t,\cdot),w_x(t,\cdot)](x) \end{aligned} \tag{9}$$
$$w(t,0) = 0 \tag{10}$$
$$w(t,1) = 0 \tag{11}$$

with initial condition:

$$w_0(x) = u_0(x) - \int_0^x k(x,y,\lambda_{\sigma(0)})u_0(y)dy \tag{12}$$

where $w : [0,T) \times [0,1] \to \mathbb{R}$ is the state of the target system. In our approach, the reaction term will be changing thanks to $\lambda_{\sigma(t)} > 0$ and according to some switching law that needs to be designed to achieve finite-time stability property. In addition, the functional $\mathcal{F}[w,w_x]$ is given as follows:

$$\mathcal{F}[w,w_x](x) = \mathcal{K}_{\sigma(t)}\left[\mathcal{L}_{\sigma(t)}[w]w_x + \mathcal{L}_{\sigma(t)}[w]\mathcal{L}^1_{\sigma(t)}[w]\right](x) \tag{13}$$

where the functional $\mathcal{L}^1_{\sigma(t)}(t)[w(t,\cdot)](x)$ is as follows:

$$\begin{aligned} \mathcal{L}^1_{\sigma(t)}(t)[w(t,\cdot)](x) = {} & l(x,x,\lambda_{\sigma(t)})w(t,x) \\ & + \int_0^x l_x(x,y,\lambda_{\sigma(t)})w(t,y)dy \end{aligned} \tag{14}$$

A suitable estimate of the functional $\mathcal{F}[w,w_x]$ given by (13) can be deduced from [11, Lemma 2].

Following the standard methodology to find kernel equations and since σ is a piecewise-constant function, it can be proved that the kernel of (7) satisfies the following PDE system:

$$k_{xx}(x,y,\lambda_{\sigma(t)}) - k_{yy}(x,y,\lambda_{\sigma(t)}) = \lambda_{\sigma(t)}k(x,y,\lambda_{\sigma(t)}) \tag{15}$$
$$k(x,0,\lambda_{\sigma(t)}) = 0 \tag{16}$$
$$k(x,x,\lambda_{\sigma(t)}) = -\tfrac{1}{2}x\lambda_{\sigma(t)} \tag{17}$$

where k is defined on the domain $\mathcal{T} = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\} \times \mathbb{R}^+$, whose solution is well-known to admit a closed form as follows [14]:

$$k(x,y,\lambda_{\sigma(t)}) = -y\lambda_{\sigma(t)}\frac{I_1\left(\sqrt{\lambda_{\sigma(t)}(x^2-y^2)}\right)}{\sqrt{\lambda_{\sigma(t)}(x^2-y^2)}} \tag{18}$$

and similarly, for the inverse transformation (7), the kernel satisfies:

$$l_{xx}(x,y,\lambda_{\sigma(t)}) - l_{yy}(x,y,\lambda_{\sigma(t)}) = -\lambda_{\sigma(t)} l(x,y,\lambda_{\sigma(t)}) \tag{19}$$

$$l(x,0,\lambda_{\sigma(t)}) = 0 \tag{20}$$

$$l(x,x,\lambda_{\sigma(t)}) = -\tfrac{1}{2} x \lambda_{\sigma(t)} \tag{21}$$

where k is defined on the domain $\mathcal{T} = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\} \times \mathbb{R}^+$, whose closed-form solution is as follows:

$$l(x,y,\lambda_{\sigma(t)}) = -y\lambda_{\sigma(t)} \frac{J_1\left(\sqrt{\lambda_{\sigma(t)}(x^2-y^2)}\right)}{\sqrt{\lambda_{\sigma(t)}(x^2-y^2)}} \tag{22}$$

In addition, from (7) and (8), one has the following relations:

$$\|w(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \leq \Psi(\lambda_{\sigma(t)}) \|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \tag{23}$$

and

$$\|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \leq \Phi(\lambda_{\sigma(t)}) \|w(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \tag{24}$$

where estimates $\Psi(\lambda_{\sigma(t)})$ and $\Phi(\lambda_{\sigma(t)})$ are given by:

$$\Psi(\lambda_{\sigma(t)}) := 1 + \left(\int_0^1 \left(\int_0^x |k(x,y,\lambda_{\sigma(t)})|^2 dy \right) dx \right)^{1/2} \tag{25}$$

with k given by (18), and

$$\Phi(\lambda_{\sigma(t)}) := 1 + \left(\int_0^1 \left(\int_0^x |l(x,y,\lambda_{\sigma(t)})|^2 dy \right) dx \right)^{1/2} \tag{26}$$

with k given by (22).

Finally, let us point out the following result [5] that is going to be used in the proof of local finite-time stability property.

Lemma 1: *For the target system* (9)-(11)*,* (13)*, there exists* c *such that the following estimate holds true:*

$$\begin{aligned} 2\left| \int_0^1 w(t,x)\mathcal{F}[w(t,\cdot), w_x(t,\cdot)](x)dx \right| \\ \leq e^{c\sqrt{\lambda_{\sigma(t)}}} (\|w(t,\cdot)\|^4_{L^2((0,1),\mathbb{R})} + \|w(t,\cdot)\|^6_{L^2((0,1),\mathbb{R})}) \\ + \|w_x(t,\cdot)\|^2_{L^2((0,1),\mathbb{R})} \end{aligned} \tag{27}$$

Proof: See [5, Section 4]. ∎

Remark 1: A precise characterization of c in (27) is not provided here; however, it can be done by analyzing the growth-in-time of the functional $\mathcal{F}$ and kernels equations involved there. The analysis may follow similar ideas of [11] which in turn follows [4].

III. State dependent switching law and Finite-time stabilization

A. On the state dependent switching law

Let us choose $\lambda_{\sigma(t)} = 2^{\sigma(t)}$ where the switching function $\sigma(t)$ is governed by the following state dependent rule[1]:

$$\sigma(t) = G(\sigma(t^-), u(t^-,\cdot)) \tag{28}$$

with $G : \mathbb{Z} \times L^2((0,1),\mathbb{R}) \to \mathbb{Z}, t^- = t + 0^-$ and $\sigma(0) \in \mathbb{Z}$.

$$G(\sigma(t), u(t,\cdot)) = \begin{cases} i+1 & \text{if} \quad \sigma(t) = i \quad \text{and} \quad \|u(t,\cdot)\|_{L^2} \leq r_{i+1} \\ i & \text{if} \quad \sigma(t) = i \quad \text{and} \quad r_{i+1} < \|u(t,\cdot)\|_{L^2} < r_{i-1} \\ i-1 & \text{if} \quad \sigma(t) = i \quad \text{and} \quad \|u(t,\cdot)\|_{L^2} \geq r_{i-1} \end{cases} \tag{29}$$

where r_i is a suitable sequence defined iteratively:

$$r_0 = \frac{e^{-\frac{c}{2}}}{\Psi(2^0)}, \quad r_i = e^{-q_i} r_{i-1}, \quad i \in \mathbb{Z} \tag{30}$$

with q_i defined as follows:

$$q_i = \ln \Psi(2^i) + \ln \Phi(2^i) + \tfrac{c}{2}\sqrt{2^i} \tag{31}$$

and Ψ and Φ are defined in (25) and (26), respectively.

Proposition 1: *Let* q_i *be defined by* (31)*. Then for* $q_i > 0$*, it holds that* $r_i \to 0$ *as* $i \to \infty$ *and*

$$\lim_{i\to\infty} \frac{q_{i+1}}{q_i} = \sqrt{2} \tag{32}$$

Proof: It follows the same lines of [18, Proposition 2]. ∎

B. On finite-time stabilization

We perform Lyapunov-based analysis on the target system and we use the bounded invertibility of the backstepping transformation in order to prove local finite-time stabilization. An estimate of the settling time is also derived.

Lemma 2: *Let us consider the target system* (9)-(12) *with* $\lambda_{\sigma(t)} = 2^{\sigma(t)}$*. Assume that* $\sigma(t) = i_k$ *for all* $t \in [t_k, t_{k+1})$ *where the switching time instants of* σ *are denoted by* $(t_k)_{k\in\mathbb{N}}$*. If* $\|w(t_k,\cdot)\|_{L^2((0,1),\mathbb{R})} < e^{-\frac{c}{2}\sqrt{2^{\sigma(t)}}}$ *and* $\theta 2^{i_k} > 1$*, then there exists a unique solution* $w \in \mathcal{C}^0\left([t_k,t_{k+1}); L^2((0,1),\mathbb{R})\right) \cap L^2_{loc}\left([t_k,t_{k+1}); H^1_0((0,1),\mathbb{R})\right)$ *such that*

$$\|w(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \leq e^{-(\theta 2^{\sigma(t)}-1)(t-t_k)} \|w(t_k,\cdot)\|_{L^2((0,1),\mathbb{R})} \tag{33}$$

for all $t \in [t_k, t_{k+1})$.

Proof:

We consider the target system (9)-(12), associated with each constant mode i_k on $[t_k, t_{k+1})$ (i.e. $\sigma(t) = i_k$) where $w(t,x) = w^{i_k}(t,x)$ for all $t \in [t_k, t_{k+1})$. That is,

[1] For hyperbolic PDEs, switching law based on Lyapunov techniques can be found for instance in [20].

$$\begin{aligned} w_t^{i_k}(t,x) &= \theta w_{xx}^{i_k}(t,x) - \theta 2^{i_k} w^{i_k}(t,x) \\ & \quad -\mathcal{F}[w^{i_k}(t,\cdot), w_x^{i_k}(t,\cdot)](x) && (34) \\ w^{i_k}(t,0) &= 0 && (35) \\ w^{i_k}(t,1) &= 0 && (36) \end{aligned}$$

The local existence and uniqueness of the solution $w \in \mathcal{C}^0\left([t_k,t_{k+1});L^2((0,1),\mathbb{R})\right) \cap L^2_{loc}\left([t_k,t_{k+1});H^1_0((0,1),\mathbb{R})\right)$ of target system (34)-(36) follow by considering the nonlinear term $\mathcal{F}[w,w_x](x)$ as a force term and using Banach fixed point theorem as proved in [5].

Let us prove now that the solution decreases exponentially on that interval. Consider the following Lyapunov function $V : L^2((0,1),\mathbb{R}) \to \mathbb{R}$, $V(w) = \int_0^1 w(x)^2 dx$. Computing the time derivative along the solutions of (34)-(36), performing integration by parts and using the boundary conditions, yield, for all $t \in [t_k, t_{k+1})$,

$$\begin{aligned} \dot{V}(t) \leq & -2\theta 2^{i_k} V(t) - 2\theta \int_0^1 w_x(x) dx \\ & + 2\Big| \int_0^1 w^{i_k}(t,x) \mathcal{F}[w^{i_k}(t,\cdot), w_x^{i_k}(t,\cdot)](x) dx \Big| \end{aligned} \tag{37}$$

with $\mathcal{F}$ is as in (13).

By Lemma 1, it holds:

$$\begin{aligned} 2\Big| & \int_0^1 w^{i_k}(t,x) \mathcal{F}[w^{i_k}(t,\cdot), w_x^{i_k}(t,\cdot)](x) dx \Big| \\ & \leq e^{c\sqrt{2^{i_k}}} (V^2(t) + V^3(t)) + \|w_x(t,\cdot)\|^2_{L^2((0,1),\mathbb{R})} \end{aligned} \tag{38}$$

for all $t \in [t_k, t_{k+1})$. Thus,

$$\dot{V}(t) \leq -2\theta 2^{i_k} V(t) + e^{c\sqrt{2^{i_k}}} (V^2(t) + V^3(t)) \tag{39}$$

Now, if $\|w^{i_k}(t_k,\cdot)\|_{L^2((0,1),\mathbb{R})} < e^{-\frac{c}{2}\sqrt{2^{i_k}}}$, we have on the one hand, $V(t) < e^{-c\sqrt{2^{i_k}}}$. Then,

$$V^2(t) < e^{-c\sqrt{2^{i_k}}} V(t) \tag{40}$$

On the other hand, since $V(t) < 1$, then $V^3(t) < V^2(t)$. Hence, from (39) and using (40), we get,

$$\dot{V}(t) \leq -2(\theta 2^{i_k} - 1) V(t) \tag{41}$$

with $\theta 2^{i_k} > 1$. Therefore, by the Comparison principle, it holds for all $t \in [t_k, t_{k+1})$, that:

$$\|w(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \leq e^{-(\theta 2^{i_k} - 1)(t - t_k)} \|w^{i_k}(t_k,\cdot)\|_{L^2((0,1),\mathbb{R})} \tag{42}$$

It concludes the proof. ■

Let us state the main result of the paper.

Theorem 1: *For any initial condition $u_0 \in L^2((0,1),\mathbb{R})$ such that*

$$\sigma(0) = i_0 \in \mathbb{Z} \quad \text{with} \quad \|u_0\|_{L^2((0,1),\mathbb{R})} \in (r_{i_0+1}, r_{i_0}] \tag{43}$$

and $\theta 2^{\sigma(0)} > 1$, then the closed-loop system (1)-(4) *with linear switched controller*

$$U(t) = \int_0^1 k(1,y,2^{\sigma(t)}) u(t,y) dy \tag{44}$$

with $k(1,y,2^{\sigma(t)})$ given in (18)*, and $\sigma(t)$ given in* (28) *according to* (29)-(30)*; has a unique solution $u \in \mathcal{C}^0\left([0,T);L^2((0,1),\mathbb{R})\right) \cap L^2_{loc}\left([0,T);H^1_0((0,1),\mathbb{R})\right)$ and is locally finite-time stable, i.e*

$$\|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \to 0, \quad \text{as} \quad t \to T \tag{45}$$

with a bounded settling time depending on initial data given by

$$T(u_0) \leq \sum_{i=i_0}^{+\infty} \frac{q_i + q_{i+1}}{\theta 2^i} + \sum_{i=i_0}^{\infty} \frac{-\frac{c}{2}\sqrt{2^i}}{\theta 2^i} < +\infty \tag{46}$$

Proof:

I. On the well-posedeness of the closed-loop solution. If $\|u(t_i,\cdot)\|_{L^2((0,1),\mathbb{R})} \in (r_{i+1}, r_i]$ and $\sigma(t) = i$ then there always exists an instance of time t^* separated from t_i such that $\|u(t^*,\cdot)\|_{L^2((0,1),\mathbb{R})} = r_{i+1}$ or $\|u(t^*,\cdot)\|_{L^2((0,1),\mathbb{R})} = r_{i-1}$. It means that σ is switched either to $i+1$ or $i-1$ according to (29) and the switching instant t^* will be always isolated (this prevents the Zeno phenomena). Then, due to local existence and uniqueness of the solution of the target system and due to the bounded invertibility of the backstepping transformation, it can be proved that for any initial condition $u_0 \in L^2((0,1),\mathbb{R})$ with (43), the closed-loop system (1)-(4), (44) has a unique solution such that $u \in \mathcal{C}^0\left([0,T);L^2((0,1),\mathbb{R})\right) \cap L^2_{loc}\left([0,T);H^1_0((0,1),\mathbb{R})\right)$, $T = \sup t_k$, where $t_k > 0$ are the sequence of switching instants.

II On the local finite-time stability. The proof for local finite-time stability is done recursively with the following induction properties, for $k \geq 0$:

- $\|u(t_k,\cdot)\|_{L^2((0,1),\mathbb{R})} \in (r_{i_k+1}, r_{i_k}]$;
- $\|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} < r_{i_k-1}$, $\forall t \in [t_k, t_{k+1})$;
- $i_k + 1 = i_{k+1}$.

Let us verify for $k = 0$, $t \in [t_0, t_1)$. From (23), it holds

$$\|w_0\|_{L^2((0,1),\mathbb{R})} \leq \Psi(2^{i_0}) \|u_0\|_{L^2((0,1),\mathbb{R})} \tag{47}$$

By initialization hypothesis (43), we get

$$\|w_0\|_{L^2((0,1),\mathbb{R})} \leq \Psi(2^{i_0}) r_{i_0} \tag{48}$$

and using the recurrence relation (30) along with (31), it holds

$$\begin{aligned} \|w_0\|_{L^2((0,1),\mathbb{R})} & \leq \Psi(2^{i_0}) e^{-q_{i_0}} r_{i_0-1} \\ & \leq \Psi(2^{i_0}) e^{-\ln(\Psi(2^{i_0})\Phi(2^{i_0}))} e^{-\frac{c}{2}\sqrt{2^{i_0}}} r_{i_0-1} \\ & \leq e^{-\ln\Phi(2^{i_0})} e^{-\frac{c}{2}\sqrt{2^{i_0}}} r_{i_0-1} \end{aligned} \tag{49}$$

Note that $e^{-\ln\Phi(2^{i_0})} r_{i_0-1} \leq 1$ by virtue of the definition of the sequence r_i in (30). Hence, we have that

$\|w_0\|_{L^2((0,1),\mathbb{R})} \leq e^{-\frac{c}{2}\sqrt{2^{i_0}}}$. Consequently, hypothesis of Lemma 2 is verified and therefore (33) holds for $t \in [t_0, t_1)$.

Then, combining (33) and (24) we have, for all $t \in [t_0, t_1)$,

$$\|u(t,\cdot)\|_{L^2} \leq \Psi(2^{i_0})\Phi(2^{i_0})e^{-(\theta 2^{i_0}-1)(t-t_0)}\|u_0\|_{L^2} \quad (50)$$

From (31), we have that (50) is rewritten as follows:

$$\|u(t,\cdot)\|_{L^2} \leq e^{q_{i_0}}e^{-\frac{c}{2}\sqrt{2^{i_0}}}e^{-(\theta 2^{i_0}-1)(t-t_0)}\|u_0\|_{L^2} \quad (51)$$

Furthermore, by initialization hypothesis (43), we get

$$\|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \leq e^{q_{i_0}}e^{-\frac{c}{2}\sqrt{2^{i_0}}}e^{-(\theta 2^{i_0}-1)(t-t_0)}r_{i_0} \quad (52)$$

Hence, using the recurrence relation (30), we get

$$\|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \leq e^{-\frac{c}{2}\sqrt{2^{i_0}}}e^{-(\theta 2^{i_0}-1)(t-t_0)}r_{i_0-1} \quad (53)$$

from which it can be deduced that $\|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} < r_{i_0-1}$. It means that norm does not reach the upper level of the threshold (preventing overshoot). In addition, it is clear, due to (53), that the norm decreases exponentially. This implies that at some time $t = t_1$ the norm will reach the lower level of threshold, i.e. $\|u(t_1,\cdot)\|_{L^2((0,1),\mathbb{R})} = r_{i_0+1}$. Therefore, by (29), the switch is done so as $\sigma(t_1) = i_0 + 1$ with $i_0 + 1 = i_{0+1} = i_1$. Hence, we can repeat the reasoning for $[t_1, t_2)$ provided the initial data $\sigma(t_1) = i_1$ with $\|u(t_1,\cdot)\|_{L^2((0,1),\mathbb{R})} \in (r_{i_1+1}, r_{i_1}]$.

By continuing the reasoning recursively, we assume that induction properties hold for $k > 0$ and we verify for $k + 1$. For that purpose it is sufficient to take as initial conditions $\sigma(t_{k+1}) = i_{k+1}$ with $\|u(t_{k+1},\cdot)\|_{L^2((0,1),\mathbb{R})} \in (r_{i_{k+1}+1}, r_{i_{k+1}}]$ and to apply the same arguments and steps as above.

It remains to prove that the settling time is bounded.

By Lemma 2 we have,

$$\|w^{i_k}(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \leq e^{-(\theta 2^{i_k}-1)(t-t_k)}\|w^{i_k}(t_k,\cdot)\|_{L^2((0,1),\mathbb{R})} \quad (54)$$

Hence, we derive,

$$t_{k+1} - t_k \leq \frac{1}{(\theta 2^{i_k}-1)}\ln\left(\frac{\|w^{i_k}(t_k,\cdot)\|_{L^2((0,1),\mathbb{R})}}{\|w^{i_k}(t_{k+1},\cdot)\|_{L^2((0,1),\mathbb{R})}}\right) \quad (55)$$

Using (23) and (24), the previous inequality is rewritten as follows:

$$t_{k+1} - t_k \leq \frac{1}{(\theta 2^{i_k}-1)}\ln\left(\frac{\Psi(2^{i_k})\Phi(2^{i_k})\|u(t_k,\cdot)\|_{L^2}}{\|u(t_{k+1},\cdot)\|_{L^2}}\right) \quad (56)$$

Taking into account $\|u(t_k,\cdot)\|_{L^2((0,1),\mathbb{R})} = r_{i_k}$ and $\|u(t_{k+1},\cdot)\|_{L^2((0,1),\mathbb{R})} = r_{i_{k+1}}$ and relation (30) with $i_k + 1 = i_{k+1}$, we get

$$t_{k+1} - t_k \leq \frac{1}{(\theta 2^{i_k}-1)}\ln\left(e^{q_{i_k+1}}\Psi(2^{i_k})\Phi(2^{i_k})\right) \quad (57)$$

Moreover, by (31) we obtain that (57) can be bounded as follows:

$$t_{k+1} - t_k \leq \frac{1}{(\theta 2^{i_k}-1)}(q_{i_k+1} + q_{i_k}) + \frac{-\frac{c}{2}\sqrt{2^{i_k}}}{(\theta 2^{i_k}-1)} \quad (58)$$

Hence,

$$T \leq \sum_{k=0}^{\infty} t_{k+1} - t_k \leq \sum_{k=0}^{\infty}\frac{q_{i_k+1}+q_{i_k}}{(\theta 2^{i_k}-1)} - \sum_{k=0}^{\infty}\frac{\frac{c}{2}\sqrt{2^{i_k}}}{(\theta 2^{i_k}-1)} \quad (59)$$

which can further be written as follows:

$$T \leq \sum_{i=i_0}^{\infty}\frac{q_{i+1}+q_i}{(\theta 2^i-1)} - \sum_{i=i_0}^{\infty}\frac{\frac{c}{2}\sqrt{2^i}}{(\theta 2^i-1)} \quad (60)$$

It can be proved that the first sum in (60) can be rewritten as

$$\sum_{i=i_0}^{\infty}\frac{q_{i+1}+q_i}{(\theta 2^i-1)} = \sum_{i=i_0}^{\infty}\frac{q_i}{(\theta 2^i-1)}\left(\frac{3\theta 2^i-4}{(\theta 2^i-2)}\right) - \frac{q_{i_0}}{(\theta 2^{i_0}-1)}\frac{2(\theta 2^{i_0}-1)}{(\theta 2^{i_0}-2)}$$

Applying the ratio test for convergence to the above series, and due to Proposition 1, we have that

$$\frac{(\theta 2^i-1)(3\theta 2^{i+1}-4)(\theta 2^i-2)}{(\theta 2^{i+1}-1)(\theta 2^{i+1}-2)(3\theta 2^i-4)}\frac{q_{i+1}}{q_i} \to \frac{\sqrt{2}}{2} < 1 \quad (61)$$

and $\frac{q_i}{(\theta 2^i-1)}\left(\frac{3\theta 2^i-4}{(\theta 2^i-2)}\right) \to 0$ as $i \to \infty$. For the second sum, the ratio test guarantees its convergence since we have

$$\frac{(\theta 2^i-1)}{(\theta 2^{i+1}-1)}\frac{\sqrt{2^{i+1}}}{\sqrt{2^i}} \to \frac{\sqrt{2}}{2} < 1, \quad \text{as} \quad i \to \infty \quad (62)$$

and $\frac{\sqrt{2^i}}{(\theta 2^i-1)} \to 0$ as $i \to \infty$.

Therefore, $T(u_0) < \infty$ for $\|u_0\|_{L^2((0,1),\mathbb{R})} \in (r_{i_0+1}, r_{i_0}]$. Hence, we finally conclude that $\|u(t,\cdot)\|_{L^2((0,1),\mathbb{R})} \to 0$ as $t \to T$. ■

IV. SIMULATIONS

We illustrate the results by considering the viscous Burgers equation (1)-(4) with $\theta = 1.1$. For numerical simulations, we implement a Crank-Nicholson scheme. We consider two different initial conditions, $u_0 = 0.1x(x-1)$ and $u_0 = x(1-x)$ satisfying (43) and we select $i_0 = 0$ with threshold sequence (30) and $c = 1$. Figure 1 shows the evolution of the L^2-norm of the closed-loop system with boundary switched linear feedback $U(t)$ given in (44) with kernel gain (18) as well as the evolution of the L^2-norm of the closed-loop system with a linear control feedback (as in e.g. [21]) for exponential stabilization. The plots are in logarithmic scale to illustrate the convergence to zero in finite-time. It can be observed that settling times are indeed dependent on initial data.

V. CONCLUSION

In this paper we have addressed the problem of local finite-time stabilization of the viscous Burgers equation under a boundary linear switched control provided the norm of initial condition is small enough. The design has been carried out based on the backstepping approach for which the switching law depends on the state of the system and kernels of

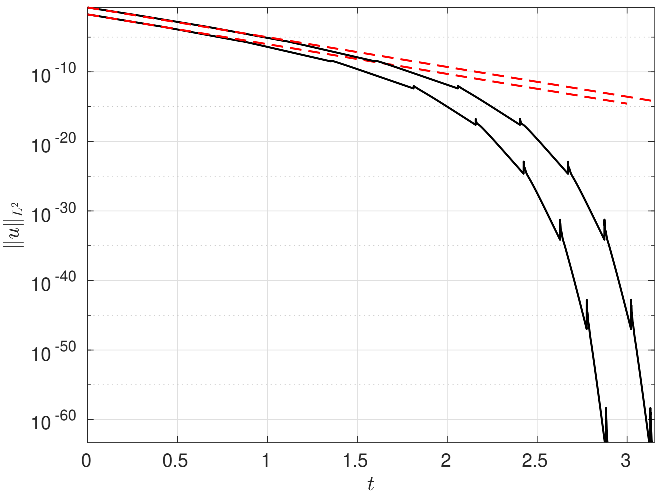

Fig. 1. Evolution of the L^2-norms of the closed-loop system (logarithmic scale) with the linear switched controller (black lines) and with linear continuous controller (red dashed lines) for exponential stabilization.

the backstepping transformation turn out to be time-varying but discontinuous (i.e. piecewise-continuous). In addition, a rigorous characterization of c involved in (27) still has to be done. This issue is then left for a future work.

Future work also includes local fixed-time stabilization of the viscous Burgers equation by means of continuous time-varying feedbacks. For that, we expect to apply our recent results [7], [8] and [6] on fixed-time stabilization of reaction-diffusion PDEs where closed-form time-varying kernels are obtained and a qualitative analysis of the fixed-time convergence is carried out. We expect also to address finite/fixed-time estimation for the viscous Burgers equation.

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